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AN OVERVIEW OF THE PHYSICAL INTERPRETATION OF DEFORMATION MEASUREMENTS

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Abstract

The physical interpretation of deformations is performed to give information on the state of internal stresses in a deformable body and on the load-deformation relationship. The latter may be obtained using either a statistical method which analysis correlation between observed deformations and observed loads or a deterministic method which utilizes information on the loads, properties of the materials, and physical laws governing the stress-strain relationship. Both methods lead to development of prediction models of deformations. The deterministic analysis must utilize numerical methods because direct solutions may be difficult or impossible to obtain. Among the numerical methods, the finite element analysis has become the most powerful. It requires sophisticated computer programs and extensive expertise in their application. The best results of the physical interpretation are obtained by combining the deterministic and statistical methods of the analysis in which the deterministic model is "calibrated" through a comparison with the statistical (empirical) model and with the actually observed deformations.

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1. Introduction

As already discussed in the preceding presentation on the geometrical analysis, a deformation survey is to serve one, or both, of two main purposes:

(1) to give information on the geometrical status of a deformable body, the change in its position, shape, and dimensions;

(2) to give information on the physical status of a deformable body, the state of internal stresses, and the load-deformation relationship.

In the first case, information on the acting forces and stresses and on the mechanical properties of the body are of no interest to the interpreter or are unavailable. The process of transforming the deformation measurements into the geometrical status has already been discussed in the previous section as the geometrical analysis. From the outcome of the geometrical analysis, one may make a qualitative interpretation of the causes of the deformation.

In the second case, the process of deriving information on the load-deformation relation is called physical interpretation. Somewhat schematically, one can perform such interpretation by using either of the two methods:

(1) A statistical method (regression analysis) which analyses the correlations between observed deformations (e.g., displacements) and observed loads (external and internal causes producing the deformation). These correlations can be obtained by performing statistical analyses on the past data. Therefore, this method is of an a posteriori nature.

(2) A deterministic method which utilizes information on the loads, properties of the materials, geometry of the body, and physical laws governing the stress-strain relationship. In contrast to the previous method, this one is of an a priori nature.

The distinction between the two methods should not be taken as absolutely clear cut. In fact, each method includes a statistical and a deterministic part. The possible forms of the model sought under (1), relating the causative quantities to the response effects, are obtained by qualitative knowledge about the expected behaviour of the body. The model determined by the deterministic method may be further enhanced through the statistical method, for instance, calibration of the physical parameters of the material from the measured deformation quantities.

The deterministic method provides the expected deformation from the measured causative quantities. If the difference between the expected deformation and the measured one is small, compared with the various errors and uncertainties which characterize the process, then the body behaves as expected and the deterministic model is justified.

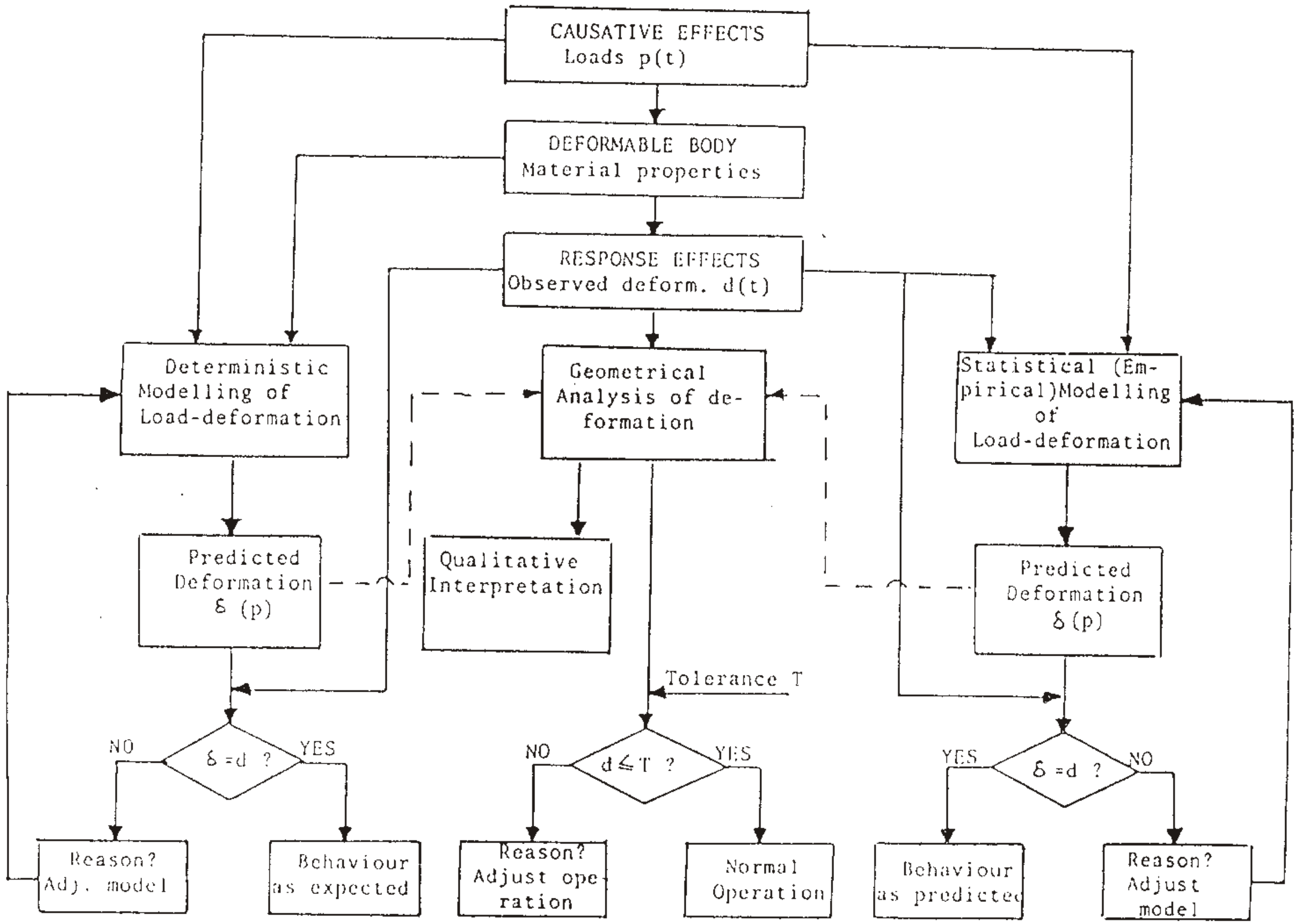


FIG.1. Flowchart of the Deformation Interpretation

Otherwise, a search for the reasons for the large discrepancies should be undertaken, and the model should be improved by combining, for instance, the deterministic and statistical models.

The statistical method establishes an empirical prediction model. Using this model, the forecasted deformation can be obtained from the measured causative quantities. A good agreement between the forecasts and the measurements then tells us that the deformable body behaves as in the past. Otherwise, as in the previous case, reasons should be found and the model should be refined.

The flowchart in Figure 1 summarizes the interpretation methods and their interactions.

2. Interpretation by Statistical Method

Interpretation by statistical method always requires a suitable amount of observations, both of causative quantities and of response effects. Let **d** be the vector of response effects. Then a functional relation between the "causes" and the "responses" can be established in the form:

$$\mathbf{d} = \mathbf{B}\,\mathbf{c} + \mathbf{v} \ , \qquad (1)$$

where **d** may be directly observed or the outcome of the geometrical analysis; **v** is the vector of errors. In the contrast to geometrical analysis, the elements of the matrix **B** are functions of the causative quantities.

Different causative quantities may produce the deformations in different ways. Some effects can be approximated by polynomial functions, but others may be more adequately expressed as trigonometric functions, and so forth. All that is embedded in the matrix **B**. The vector **c** in eqn. (1), representing the magnitude of the effects, is to be estimated.

Let us take as an example modelling of the response of a power dam to the causative effects as a function to time t. The horizontal displacement $d_i(t)$ of a point i can be modelled as:

$$d_i(t) = F_i(t) + L_i(t) + G_i(t) + v(t) \ , \qquad (2)$$

where $F_i(t)$ is the hydrostatic pressure component, $L_i(t)$ is the thermal component, $G_i(t)$ is the irreversible component due to the non-elastic behaviour of the dam, and v(t) is the error component. The hydrostatic pressure component is a function of the water elevation h(t) in the reservoir:

$$F_i(t) = a_{io} + a_{i1}h(t) + a_{i2}h^2(t) + \ldots + a_{im}h^m(t) \ . \qquad (3)$$

The thermal component can be modelled in two ways. If some key temperatures $T_i(t)$, (i=1,...,k) in the dam are measured, then

$$L_i(t) = e_{i1}T_1(t) + e_{i2}T_2(t) + \dots + e_{ik}T_k(t) \,. \tag{4}$$

If no temperature measurements are available, then

$$L_i(t) = f_{i1}\sin\omega t + c_{i1}\cos\omega t + f_{i2}\sin 2\omega t + c_{i2}\cos 2\omega t + \dots + f_{ip}\sin p\omega t + c_{ip}\cos p\omega t \,. \tag{5}$$

where $\omega = 2\pi/\tau$; $\tau = 1$ year. The irreversible component originates from non-elastic phenomena, like creep of the dam, and is usually approximated by exponential functions:

$$G_i(t) = \sum_{1}^{l} g_l \, e^{-k_l(t-t_l)} \,. \tag{6}$$

In the above expressions, all the coefficients form the vector **c** in eqn. (1). The matrix **B** is established from the base functions in eqns. (3) and (4) or (5) and (6).

Applying the least-squares criterion, the vector **c** and its variance-covariance matrix are estimated from

$$\hat{\mathbf{c}} = (\mathbf{B}^T\mathbf{P}_d\mathbf{B})^{-1}\,\mathbf{B}^T\mathbf{P}_d\,\mathbf{d} \,, \tag{7}$$

and

$$\Sigma_c = \sigma_o^2(\mathbf{B}^T\,\mathbf{P}_d\,\mathbf{B})^{-1} \,, \tag{8}$$

where $\mathbf{P}_d$ is the weight matrix of **d** and σ_o^2 is the a priori variance factor. The a posteriori variance factor

$$\hat{\sigma}_o^2 = ((\mathbf{d} - \mathbf{B}\hat{\mathbf{c}})\,\mathbf{P}_d(\mathbf{d} - \mathbf{B}\hat{\mathbf{c}}))\,/\,(df) \,, \tag{9}$$

with df being the degrees of freedom, serves as an indicator of the appropriateness of the deformation response model **Bc**. If the inequality,

$$\hat{\sigma}_o^2/\sigma_o^2 \le F(\alpha;\, df,\, \infty) \,, \tag{10}$$

is satisfied, then one may accept that the deformation is satisfactorily explained by the model. If some of the estimated coefficients become statistically insignificant, then they should be excluded from the model and the model must be modified. The detailed discussion on the statistical tests can be found in Chen [1983], and it has been summarized in the section on geometrical analysis.

Similar to the procedures and methods in geometrical analysis, those in statistical testing are applicable to the statistical method of the physical interpretation. The same three basic steps are usually followed, that is, preliminary identification of the response model, estimation of unknown coefficients, and diagnostic checking of the model.

The statistical method of the physical interpretation possesses some undeniable merits:

(1) knowledge of the mechanical properties of a deformable body is not required;

(2) good results from the point of view of prediction are usually obtained.

But it also contains some undesirable features:

(1) a comparatively large amount of data on both causative and response quantities is needed in order to obtain a reliable model;

(2) the information (prediction) cannot be generalized to other deformable bodies because the body in this case is acting only as a "black box."

Example: The displacements of a point on an arch-gravity concrete dam was measured weakly and plotted in Figure 2(b). At the same time, the water elevation in the reservoir was recorded, as shown in Figure 2(a). No temperature measurements were available. Using models (3) and (5), the following statistical model of the deformation (displacement in millimetres) has been obtained:

$$d(t) = L(t) + F(t)$$

where

$$L(t) = -6.287 \sin(\omega t) - 8.4709 \cos(\omega t) + 1.1018 \sin(2\omega t) - 0.0042 \cos(2\omega t)$$

and

$$F(t) = 30.2524 - 16.3029\, x(t) + 24.7444\, x^2(t) - 35.132\, x^3(t),$$

with

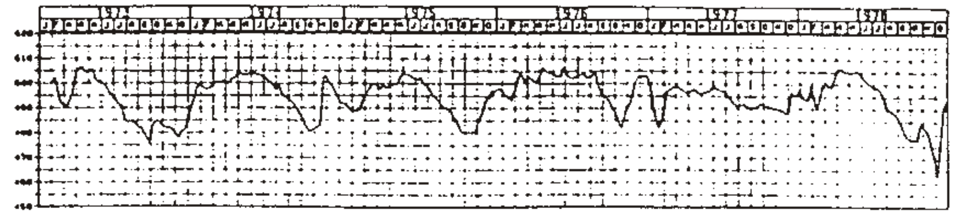
$$x(t) = [h(t) - 462.4]/44.6\ .$$

Figure 2(c) shows the differences between the measured and modelled (predicted) displacements. The differences do not indicate any time dependency and, therefore, the irreversible component (6) has not been included in the model.

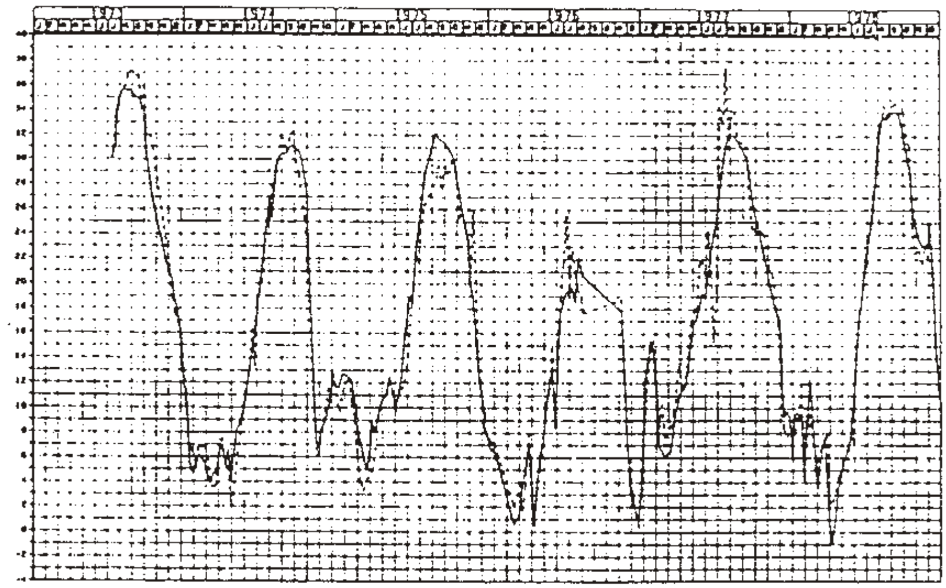
3. Interpretation by Deterministic Method

Any real material deforms if an external force is applied to it. The external forces may be of two kinds: surface forces, i.e., forces distributed over the surface of the body, such as the pressure of one body on another; and body forces, which are distributed over the volume of the body, such as gravitational forces, thermal stress or, in the case of a body in motion, inertial forces. Under the action of external forces, a state of stress exists in a body.

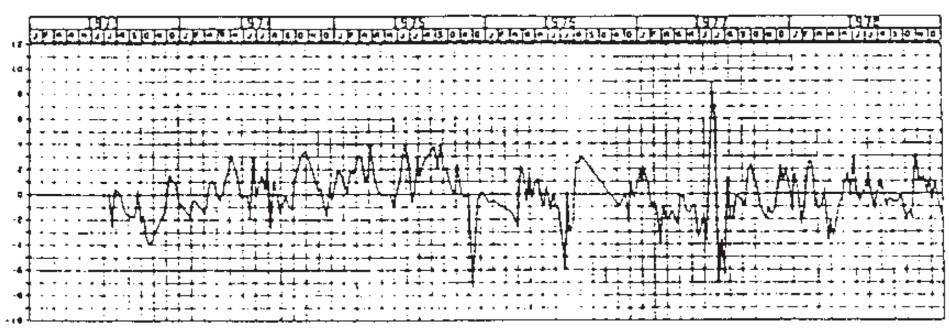
As discussed in Chrzanowski and Chen [1986], the deformation of a body is fully described if six components of the strain tensor are known at every point. The six components consist of three components of normal strain, ε_x, ε_y, and ε_z, which describe change in the dimensions at the point (x, y, z) along the x, y, and z axes of the coordinate system, and three components of the shearing strain ε_{xy}, ε_{xz}, and ε_{yz}, which describe change in the shape of the element (angular changes) in the corresponding planes of the



a) Water level in the reservoir (m)



b) Superposition of measured (-----) and statistically modelled (———) displacements (mm)



c) discrepancies between obsrved and modelled displacements in mm.

FIG.2. Comparison between measured and predicted displacements of a point on a dam structure (after ENEL,1980)

coordinate system.

Similarly, the state of stress at any point of the medium is completely characterized by the specification of six components of stress tensor: three components of normal stress, σ_x, σ_y, and σ_z; and three components of shearing stress, σ_{xy}, σ_{xz}, and σ_{yz}.

The relation between the strain tensor and stress tensor is governed by the generalized Hooke's law. For a homogeneous isotropic medium, the generalized Hooke's law can be written as:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix} = E/((1+\upsilon)(1-2\upsilon)) \begin{bmatrix} 1-\upsilon & \upsilon & \upsilon & 0 & 0 & 0 \\ \upsilon & 1-\upsilon & \upsilon & 0 & 0 & 0 \\ \upsilon & \upsilon & 1-\upsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\upsilon)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\upsilon)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\upsilon)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix} \quad (11)$$

or, more compactly, as

$$\sigma = \mathbf{D}\,\varepsilon, \quad (11')$$

where E is Young's modulus, υ is Poisson's ration, and **D** is the constitutive matrix of the material.

In order to determine the relation between external forces, state of stress, and displacements, the solution must satisfy the three basic conditions [Timoshenko and Goodier, 1970]:

(1) the equilibrium of forces (external and internal),

(2) the compatibility of displacements, and

(3) the law of material behaviour (the stress-strain relation eqn. (11)).

The compatibility condition requires that the deformed structure fits together, i.e., that deformations of its elements are compatible and no discontinuities are created in the process of the deformation. The equilibrium conditions have the form [Sokolnikoff, 1956]:

$$-\mathbf{f} = \mathbf{B}^T \cdot \sigma, \quad (12)$$

where $\mathbf{f}^T = (f_x, f_y, f_z)$ is body force and matrix **B** is a differential operator:

$$\mathbf{B}^T = \begin{bmatrix} \partial/\partial x & 0 & 0 & \partial/\partial y & 0 & \partial/\partial z \\ 0 & \partial/\partial y & 0 & \partial/\partial x & \partial/\partial z & 0 \\ 0 & 0 & \partial/\partial z & 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \quad (13)$$

Considering the strain-displacement relation

$$\varepsilon = \mathbf{B}\ \mathbf{d}\ , \tag{14}$$

where $\mathbf{d} = (u, v, w)^T$ is the displacement and combining eqns. (11'), (12), and (14), one obtains the differential equations for displacements:

$$-\mathbf{f} = \mathbf{B}^T\ \mathbf{D}\ \mathbf{B}\ \mathbf{d}\ . \tag{15}$$

In principle, when the boundary conditions, either in the form of displacements or in the form of the acting forces, are given and the body forces are prescribed, the differential eqns. (15) are completely solved. However, direct solutions may be difficult, and numerical methods have to be used where the finite element method provides a powerful tool. The finite element technique gives an exact solution to a problem which approximates the differential eqns. (15).

The basic concept of the finite element method is that the continuum of the deformable body is replaced by an assemblage of individual small elements of finite dimensions which are connected together only at the nodal points of the elements. The elements may be of any shape but usually triangular or rectangular elements are chosen for two-dimensional analysis and rectangular or trapezoidal "bricks" are used in the three-dimensional solutions. Figure 3 gives as an example a three-dimensional finite element mesh of a dam and its foundation used for computation of displacements of points on the dam [ENEL, 1980]. Figure 4 is a two-dimensional mesh for predicting ground movements due to mining activities [Chrzanowski and Szostak-Chrzanowski, 1985]. For each element, one can establish the relationship between the nodal forces and displacements. For example, in the two-dimensional analysis with triangular elements, the displacement at point (x, y) can be modelled as

$$u = a_1 + a_2x + a_3y$$
$$v = a_4 + a_5x + a_6y\ , \tag{16}$$

where u, v are the components of displacement in x and y directions, respectively.

Since both components are linear in x and y, the displacement continuity between the adjoining elements for any nodal displacement is ensured. Thus the finite element model of a deformable body involves a piecewise polynomial interpolation of displacement field. The nodal displacements define several displacement fields that are laid side by side. Let $\mathbf{d}_e$ be a vector of displacements for the three nodal points, i.e., $\mathbf{d}_e^T = (u_1, v_1, u_2, v_2, u_3, v_3)$ and $\mathbf{a}^T = (a_1, ..., a_6)$. Then, for an element, the expression (16) can be written as

$$\mathbf{d}_e = \mathbf{M}_e\ \mathbf{a}\ , \tag{17}$$

with $\mathbf{M}_e$ being a 6 by 6 matrix, easily obtained from eqn. (16). Applying the relation (14), we obtain the strain vector

$$\varepsilon^T = (\varepsilon_x, \varepsilon_y, \varepsilon_{xy}) = (a_2, a_6, a_3 + a_5)\ ,$$

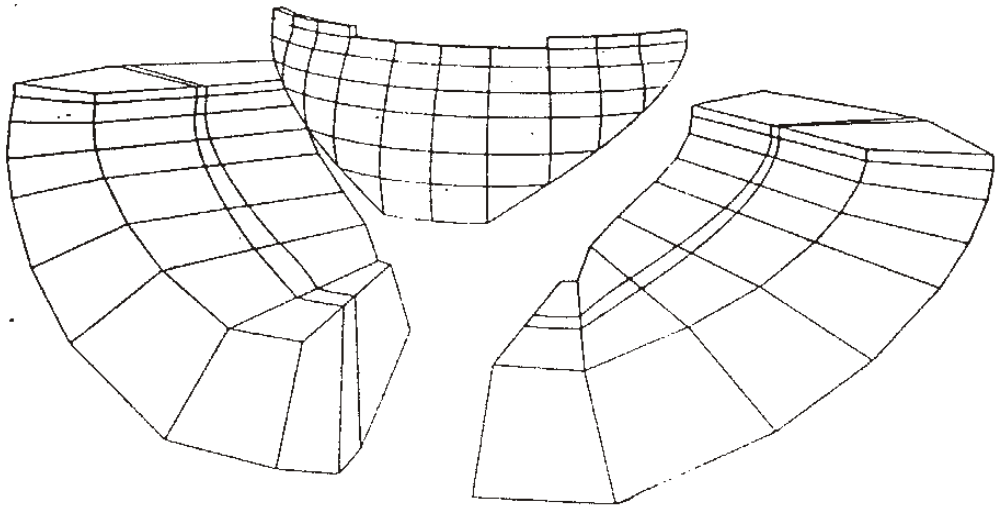


FIG.3. Three-dimensional finite element mesh of a dam
and its foundation (after ENEL,1980)

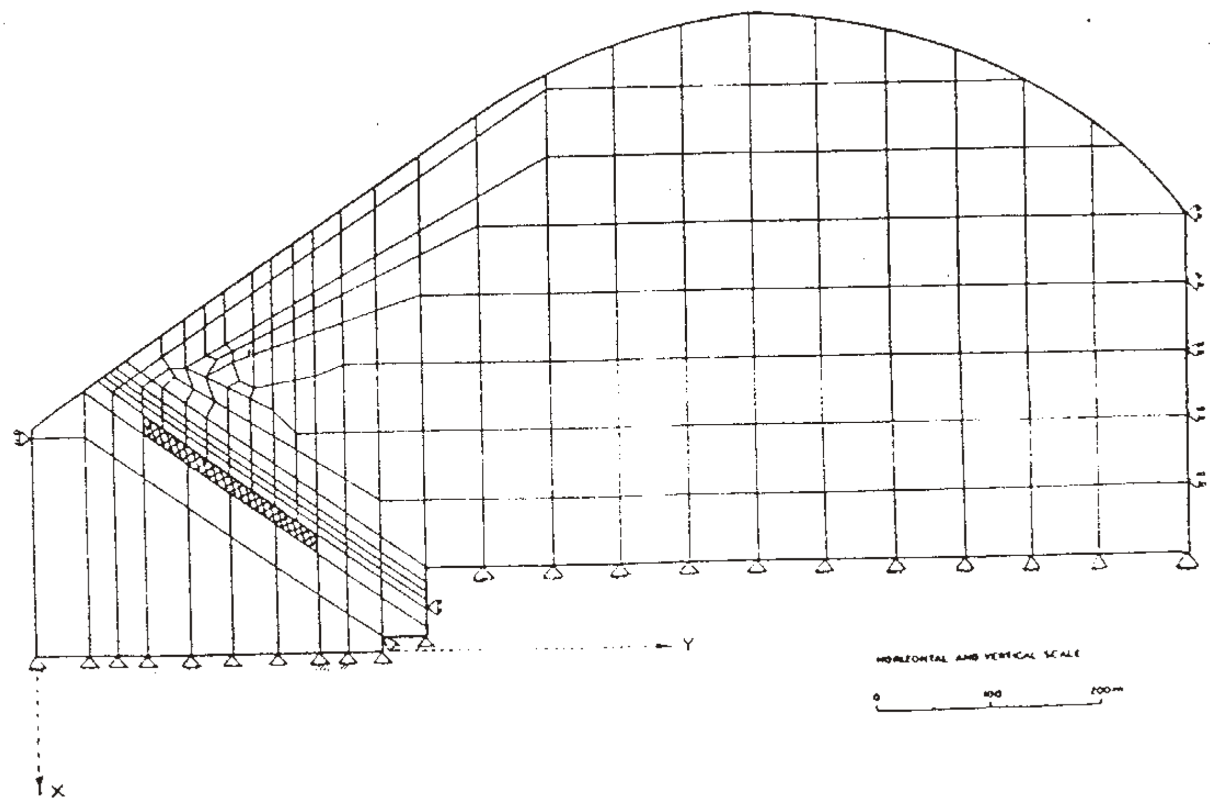


FIG.4. Finite element mesh for ground subsidence analysis in
a mining area (after (Chrzanowski and Szostak-Chrzanowski,1985)

or, in shorthand matrix notation,

$$\varepsilon = \mathbf{N}\,\mathbf{a}\,. \tag{18}$$

Combining eqns. (17) and (18), the strain-displacement relation may be expressed briefly in the form:

$$\varepsilon = \mathbf{N}\,\mathbf{M}_e^{-1}\,\mathbf{d}_e = \widetilde{\mathbf{B}}\,\mathbf{d}_e\,. \tag{19}$$

Considering eqn. (11), the internal stresses are related to nodal displacements by

$$\sigma = \mathbf{D}\,\widetilde{\mathbf{B}}\,\mathbf{d}_e\,. \tag{20}$$

Furthermore, the internal stresses are replaced with statistically equivalent nodal forces $\mathbf{f}_e$, resulting in the relationship between the nodal forces and displacements [Rockey et al., 1975]:

$$\begin{aligned}\mathbf{f}_e &= [\textstyle\int_v \widetilde{\mathbf{B}}^T\,\mathbf{D}\,\widetilde{\mathbf{B}}\,dv]\cdot\mathbf{d}_e \\ &= \mathbf{k}_e\,\mathbf{d}_e\,,\end{aligned} \tag{21}$$

where $\mathbf{k}_e$ is called the stiffness matrix of the element. Comparing eqn. (21) with eqn. (15), one can recognize that the key step in the finite element method is the approximation of the differential operator **B** by a linear algebraic operator.

Once the stiffness matrices for all elements of the deformable body have been calculated an overall structural stiffness matrix **K** is composed by a superposition of the stiffness matrices for all the elements, and the total equilibrium equation for the whole body is written:

$$\widetilde{\mathbf{f}} = \mathbf{K}\cdot\widetilde{\mathbf{d}}\;, \tag{22}$$

where $\widetilde{\mathbf{f}}$ is a vector of applied nodal loads in the whole body, and $\widetilde{\mathbf{d}}$ is a vector of nodal displacements. If boundary conditions are known, then forces at any nodal points can be solved from eqn. (22).

If initial stresses and strains in the material are known and the behaviour of the material (strain-stress relationship) follows a linear elastic model, then the finite element analysis may give a good prediction of deformations. It applies, for instance, to most man-made structures made of steel or concrete in which the properties of the material and acting forces can be determined with a high accuracy. However, in most geotechnical and rock mechanics studies the behaviour of materials, such as soil or in-situ rocks, must be modelled by non-linear elastic or plastic behaviour and then the use of the finite element analysis, which requires extensive experience, must be treated with extreme caution. Though some sophisticated computer programs, such as FEMMA (Finite Element Analysis for Mining Applications) which has recently been developed at UNB (Szostak-Chrzanowski, unpublished), can handle the non-linear behaviour in ground subsidence studies [Chrzanowski and Szostak-Chrzanowski, 1986], still more research is needed in this area.

4. Interpretation by a Combination of the Deterministic and Statistical Methods

Due to many uncertainties in the deterministic model of deformations, the theoretically calculated displacements δ (or any other deformation quantities) will generally depart from the observed values $\mathbf{d}$ by Δ, i.e.,

$$\Delta = \delta - \mathbf{d} \quad . \tag{23}$$

The discrepancies may be due to:

- imperfect knowledge of the material properties, for example, errors in the elasticity constants;
- aforementioned wrong modelling of the behaviour (elastic instead of plastic or creep neglected, etc.) of the material;
- errors in the thermal parameters of the material;
- approximation in calculations;
- measuring errors in $\mathbf{d}$;
- measuring errors in loading (causative) effects; and so on.

Investigation of the discrepancies is useful in gaining a better knowledge of the behaviour of the deformable body. Let Σ_Δ, $\Sigma_\mathbf{d}$, and Σ_δ be their covariance matrices, then

$$\Sigma_\Delta = \Sigma_\mathbf{d} + \Sigma_\delta \quad . \tag{24}$$

In order to test whether the discrepancies Δ are of a systematic nature, we have a null hypothesis $H_o : \Delta = \mathbf{0}$ against an alternative hypothesis $H_A : \Delta \neq \mathbf{0}$. The test statistic is

$$T = \Delta^T \Sigma_\Delta^{-1} \Delta \tag{25}$$

with a critical value: $\chi^2(\alpha;\, df)$, where df is the rank of Σ_Δ (degrees of freedom). If $T > \chi^2(\alpha;\, df)$ at the $(1 - \alpha)\%$ level of confidence, then H_o is rejected and a further search for an explanation of the discrepancies is required. At this stage, the deterministic and statistical methods are combined for the interpretation of the deformation measurements.

Assume that the systematic discrepancies are caused, for example, by the improperly chosen material parameters. Then, new ("calibrated") values of the parameters can be estimated by applying the principle of least squares:

$$\min\{(\delta - \mathbf{d})^T \Sigma_\Delta^{-1}(\delta - \mathbf{d})\} \quad . \tag{26}$$

For example, the temperature induced component of the displacement of a point i on a concrete dam is proportional to the thermal expansion coefficient α, and the hydrostatic pressure component is inversely proportional to Young's modulus E of concrete. Therefore, the discrepancy $\Delta_i(t)$ is modelled as

$$\Delta_i(t) = (\alpha/\bar{\alpha})\ \bar{L}_i(t) + (\bar{E}/E)\ \bar{F}_i(t) - d_i(t) \tag{27}$$

with two components $\bar{L}_i(t)$ and $\bar{F}_i(t)$ being calculated from the deterministic model using the values of $\bar{\alpha}$ and $\bar{E}$. Applying the least-squares principle (26), the "calibrated" values of E and α are estimated.

This method for calibration of the constants of the material properties, however, may lead to physically unacceptable values of the calibrated quantities if the discrepancy is of a different source. In such a case, one has to try another approach to the interpretation by using the statistical analysis of the discrepancies, as discussed above in section 2, in order to find the reason and then enhance the deterministic model.

5. Concluding Remarks

Physical interpretation of deformation surveys requires interdisciplinary knowledge. Surveying engineers have a good knowledge of data acquisition and other specialists, e.g., geophysicists and civil engineers, are well acquainted with the behaviour of the deformable body. Efficient cooperation between them is indispensable in order to successfully interpret the measured results.

Surveying engineers and scientists have not been very involved in the deformation interpretation which has usually been done by other specialists. This situation should be changed. Their involvement would contribute to the interpretation of deformation surveys. In addition, by participating in the interpretation of deformation surveys, surveyors would gain a good insight into deformation measurements, which would contribute to the optimal design of monitoring schemes.

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